$$\frac{1}{X+a} \qquad \sum \frac{1}{X+a_i} \qquad \frac{1}{X+b}$$

Partial Fraction Techniques For Cryptography

$$\frac{1}{X+a} \cdot \frac{1}{X+b} = \frac{1}{b-a}\left(\frac{1}{X+a} - \frac{1}{X+b}\right)$$

$$\mathscr{C} = \begin{bmatrix} \frac{1}{a_1-b_1} & \frac{1}{a_1-b_2} & \cdots & \frac{1}{a_1-b_n} \\ \frac{1}{a_2-b_1} & \frac{1}{a_2-b_2} & \cdots & \frac{1}{a_2-b_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_m-b_1} & \frac{1}{a_m-b_2} & \cdots & \frac{1}{a_m-b_n} \end{bmatrix}$$

Rohit Nema (Stanford) | Bay Area Crypto Day | Nov. 14, 2025
joint work with Arnab Roy (Mysten Labs) and Charanjit Jutla (IBM)

Summary

- We use Partial Fraction Decomposition (of rational functions with linear denominator) to do cryptography

$$\frac{1}{X+c},\ c \in \mathbb{F}_p$$

- Not the first to use rational functions (e.g. [BB04b,DY05,JL09,Wee16,Hab22,GV22])
- Two interesting properties: (1) *membership testing* and (2) *linear independence*

$$(1) \Longrightarrow \textbf{Key-Value Commitments} \qquad (2) \Longrightarrow \textbf{Dynamic Threshold Encryption}$$

- Prove security using new (*adaptive)* falsifiable q-type assumptions (secure in the (bilinear) generic group model)

Partial Fraction Decomposition

the mathematical foundation of our work

- rewrite product of rational functions as a sum (with easily computable closed-form coefficients)

$$\frac{1}{X+3} \cdot \frac{1}{X+5} = \frac{1}{2}\left(\frac{1}{X+3} - \frac{1}{X+5}\right)$$

well-known

Theorem. Suppose $n \geq 2$. Let $a_1, \ldots, a_n \in \mathbb{F}$ be distinct. Then, there exists unique coefficients $c_1, \ldots, c_n \in \mathbb{F}$ such that,

$$\prod_{i \in [n]} \frac{1}{X + a_i} = \sum_{i \in [n]} \frac{c_i}{X + a_i}$$

where $c_i = \prod_{j \neq i} \frac{1}{a_j - a_i}$ for each $i \in [n]$.

Partial Fraction Decomposition

(Non-)Membership Testing (1)

- Represent set $S = \{a_1, \ldots, a_n\}$ as a sum of fractions, $\displaystyle\sum_{i \in [n]} \frac{1}{X + a_i}$

- For $b \in \mathbb{F}$, examine the product, $P = \left(\displaystyle\sum_{i \in [n]} \frac{1}{X + a_i} \right) \left(\frac{1}{X + b} \right)$
  - If $b \notin S$, then there exists h_i such that, $P = \displaystyle\sum_{i \in [n]} \frac{z_i}{X + a_i} + \frac{z_{n+1}}{X + b}$
  - If $b \in S$, then no such decomposition exists

Partial Fraction Decomposition

Linear Independence (2)

- Consider again,

$$\left(\sum_{i\in[n]}\frac{1}{X+a_i}\right)\left(\frac{1}{X+b}\right)=\sum_{i\in[n]}\frac{z_i}{X+a_i}+\frac{z_{n+1}}{X+b}$$

- Using **Theorem 1**, it is easy to show that,

$$z_i=\frac{1}{b-a_i}\text{ and }z_{n+1}=-\sum_{i\in[n]}\frac{1}{b-a_i}$$

- For $m\leq n$ distinct $b_1,\ldots,b_m\notin S$, the following are $\mathbb{F}$-linearly independent,

$$\left\{\left(\sum_{i\in[n]}\frac{1}{X+a_i}+\sum_{i\in[n]}\frac{1}{b_j-a_i}\right)\left(\frac{1}{X+b_j}\right)\right\}_{j\in[m]}=\left\{\sum_{i\in[n]}\frac{1}{b_j-a_i}\cdot\frac{1}{X+a_i}\right\}_{j\in[m]}$$

Partial Fraction Decomposition

Linear Independence (2)

- For $m \leq n$ distinct $b_1, \ldots, b_m \notin S$, the following are $\mathbb{F}$-linearly independent,

$$\left\{ \left(\sum_{i \in [n]} \frac{1}{X + a_i} \right) \left(\frac{1}{X + b_j} \right) + \sum_{i \in [n]} \frac{1}{b_j - a_i} \right\}_{j \in [m]} = \left\{ \sum_{i \in [n]} \frac{1}{b_j - a_i} \cdot \frac{1}{X + a_i} \right\}_{j \in [m]}$$

- **Why?**
  - the coefficients $\left(\frac{1}{b_j - a_i} \right)$ form a full-rank C*auchy* matrix*!*

Cauchy Matrices

- For $\vec{a} \in \mathbb{F}^n, \vec{b} \in \mathbb{F}^m$, we define the Cauchy matrix $\mathscr{C}(\vec{a}, \vec{b}) = \left(c_{ij}\right)_{i \in [n], j \in [m]}$ as,

$$c_{ij} = \frac{1}{b_j - a_i}$$

- So,

$$\mathscr{C}(\vec{a}, \vec{b}) = \begin{bmatrix} \frac{1}{b_1 - a_1} & \frac{1}{b_2 - a_1} & \cdots & \frac{1}{b_m - a_1} \\ \frac{1}{b_1 - a_2} & \frac{1}{b_2 - a_2} & \cdots & \frac{1}{b_m - a_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{b_1 - a_n} & \frac{1}{b_2 - a_n} & \cdots & \frac{1}{b_m - a_n} \end{bmatrix}$$

one can show this is full-rank

if $(a_1, \ldots, a_n, b_1, \ldots, b_m)$ are all distinct

Product of Partial Fractions as Cauchy Matrix

$$\left\{\left(\sum_{i\in[n]}\frac{1}{X+a_i}\right)\left(\frac{1}{X+b_j}\right)+\sum_{i\in[n]}\frac{1}{b_j-a_i}\right\}_{j\in[m]}=\left\{\sum_{i\in[n]}\frac{1}{b_j-a_i}\cdot\frac{1}{X+a_i}\right\}_{j\in[m]}$$

$$\mathscr{C}(\vec{a},\vec{b})^T\begin{bmatrix}\frac{1}{X+a_1}\\ \vdots\\ \frac{1}{X+a_n}\end{bmatrix}=\begin{bmatrix}\frac{1}{b_1-a_1} & \frac{1}{b_1-a_2} & \cdots & \frac{1}{b_1-a_n}\\ \frac{1}{b_2-a_1} & \frac{1}{b_2-a_2} & \cdots & \frac{1}{b_2-a_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{b_m-a_1} & \frac{1}{b_m-a_2} & \cdots & \frac{1}{b_m-a_n}\end{bmatrix}\begin{bmatrix}\frac{1}{X+a_1}\\ \vdots\\ \frac{1}{X+a_n}\end{bmatrix}$$

Product of Partial Fractions as Cauchy Matrix

A 2 x 2 Example

Let $n = 2,\ m = 2.$

Suppose $a_1 = 1, a_2 = 4, b_1 = 3,$ and $b_2 = 8.$

$$\left(\frac{1}{X+1}+\frac{1}{X+4}\right)\left(\frac{1}{X+3}\right)+\left(\frac{1}{3-1}+\frac{1}{3-4}\right)\frac{1}{X+3}=\frac{1}{3-1}\cdot\frac{1}{X+1}+\frac{1}{3-4}\cdot\frac{1}{X+4}$$

$$\left(\frac{1}{X+1}+\frac{1}{X+4}\right)\left(\frac{1}{X+8}\right)+\left(\frac{1}{8-1}+\frac{1}{8-4}\right)\frac{1}{X+8}=\frac{1}{8-1}\cdot\frac{1}{X+1}+\frac{1}{8-4}\cdot\frac{1}{X+4}$$

$$\begin{bmatrix} \frac{1}{3-1} & \frac{1}{3-4} \\ \frac{1}{8-1} & \frac{1}{8-4} \end{bmatrix} \begin{bmatrix} \frac{1}{X+1} \\ \frac{1}{X+4} \end{bmatrix}$$

Partial Fraction Decomposition

Recap: Linear Independence (2)

- Consider again,

$$\left(\sum_{i\in[n]} \frac{1}{X+a_i}\right)\left(\frac{1}{X+b}\right) = \sum_{i\in[n]} \frac{z_i}{X+a_i} + \frac{z_{n+1}}{X+b}$$

- Using **Theorem 1**, it is easy to show that,

$$z_i = \frac{1}{b-a_i} \text{ and } z_{n+1} = -\sum_{i\in[n]} \frac{1}{b-a_i}$$

- For $m \leq n$ distinct $b_1, \ldots, b_m \notin S$, the following are $\mathbb{F}$-linearly independent,

$$\left\{\left(\sum_{i\in[n]} \frac{1}{X+a_i} + \sum_{i\in[n]} \frac{1}{b_j-a_i}\right)\left(\frac{1}{X+b_j}\right)\right\}_{j\in[m]} = \left\{\sum_{i\in[n]} \frac{1}{b_j-a_i}\cdot\frac{1}{X+a_i}\right\}_{j\in[m]}$$

First Construction

Key-Value Commitments

Key-Value Commitments (KVCs) [AR20]

using (Non-)Membership Testing (1)

- We have a dictionary, a set of key-value pairs,

$$D = \left\{ \left(k_i, v_i \right) \right\}_{i \in [n]}$$

- *Commit* using "linear" partial fractions,

$$C_D(X) = \sum_{i \in [n]} \frac{v_i}{X + k_i}$$

- We can test,

$$\frac{(k, v) \in D}{\begin{array}{c}\exists z_i \text{ such that,} \\ \left(C_D(X) - \frac{v}{X + k} \right) \left(\frac{1}{X + k} \right) = \sum_{i \in [n]} \frac{z_i}{X + k_i}\end{array}} \qquad \frac{(k, v) \notin D}{\begin{array}{c}\textit{no such decomposition exists} \\ \text{because of quadratic fraction,} \\ \frac{1}{(X + k)^2}\end{array}}$$

Key-Value Commitments (KVCs)

Instantiating Cryptographically

- Asymmetric bilinear group, $\mathscr{G} = \left(p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, g_1, g_2, e\right)$,
  - Additive notation: $[x]_i = g_i^x \in \mathbb{G}_i$, and $[y]_T = e([1]_1, [1]_2)^y \in \mathbb{G}_T$
- Trusted setup/DKG: sample $x \xleftarrow{\$} \mathbb{F}$,
- Let $K_i(X) = \dfrac{1}{X + k_i}$. Then, commit to dictionary D using,

additively homomorphic!
efficient to update

$$[C_D(x)]_1 = \left[\sum_{i \in [n]} \frac{v_i}{x + k_i}\right]_1 = \sum_{i \in [n]} v_i \left[K_i(x)\right]_1$$

- Membership test is a pairing equation,

where does verifier get these?

$$e\left([C_D(x)]_1 - v\left[\frac{1}{x+k}\right]_1, \left[\frac{1}{x+k}\right]_2\right) \stackrel{?}{=} e\left(\sum_{i \in [n]} z_i \left[K_i(x)\right]_1, [1]_2\right)$$

linear in $|D|$ to compute :(

Credential-based KVCs

- New *semi-trusted* party, the *Registrar*: knows x and D and runs in $O(|D|)$ time.
- Takes as input k_{n+1} and outputs a *credential*,

$$h_{n+1}^{(1)} = \left[\frac{1}{x + k_{n+1}}\right]_1, h_{n+1}^{(2)} = \left[\frac{1}{x + k_{n+1}}\right]_2, \Lambda = \sum_{i \in [n+1]} \left[\frac{z_i}{x + k_i}\right]_1$$

- *Semi-stateless* setting: stateless and $O(1)$ to update commitment after one-time linear computation of credential
  - after each update, user publishes *update information* and every user updates their proofs

Updating commitment

$$\left[C_D(x)\right]_1 + v_{n+1} h_{n+1}^{(1)}$$

Update information

$$\left(k_{n+1}, v_{n+1}, h_{n+1}^{(1)}\right)$$

Updating proof for user i

$$\Lambda + \frac{v_{n+1}}{k_{n+1} - k_i}\left(h_i^{(1)} - h_{n+1}^{(1)}\right)$$

$$+$$

verify $h_{n+1}^{(1)}$ is well-formed

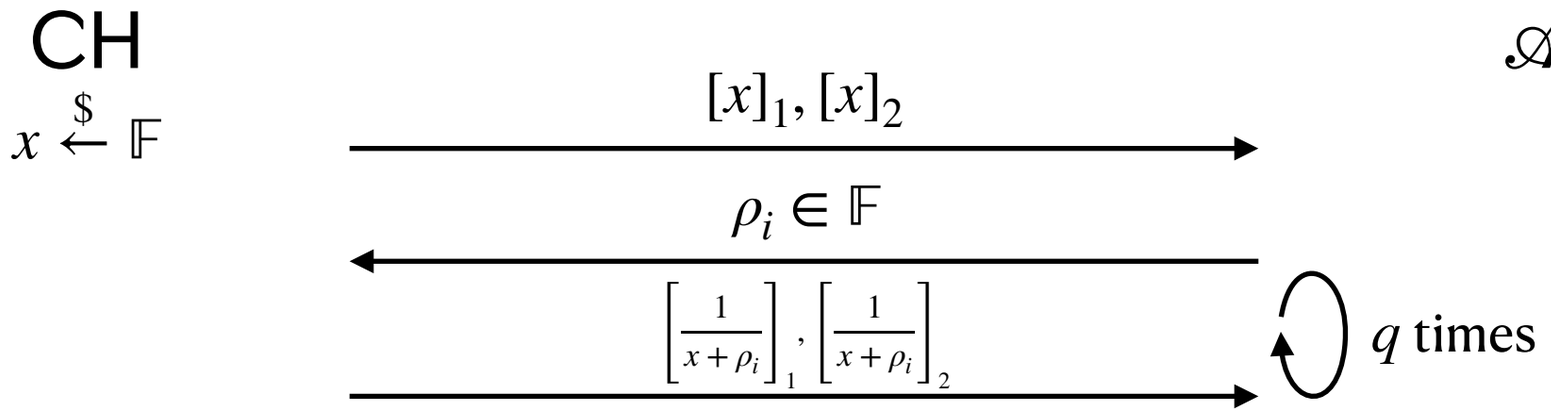
$$e\left(h_{n+1}^{(1)}, [x]_2 + [k_{n+1}]_2\right) \stackrel{?}{=} e([1]_1, [1]_2)$$

Credential-based KVCs

Defining and Proving Security

- Usual notion of *key-binding* (slightly modified to account for *unregistered* keys)
- Two new assumptions, (1) q-DistPole: reminiscent of BB signature *unforgeability,* and (2) q-SqPole: cannot compute a quadratic fraction from linear fractions

q-DistPole



$\mathscr{A}$ wins if,

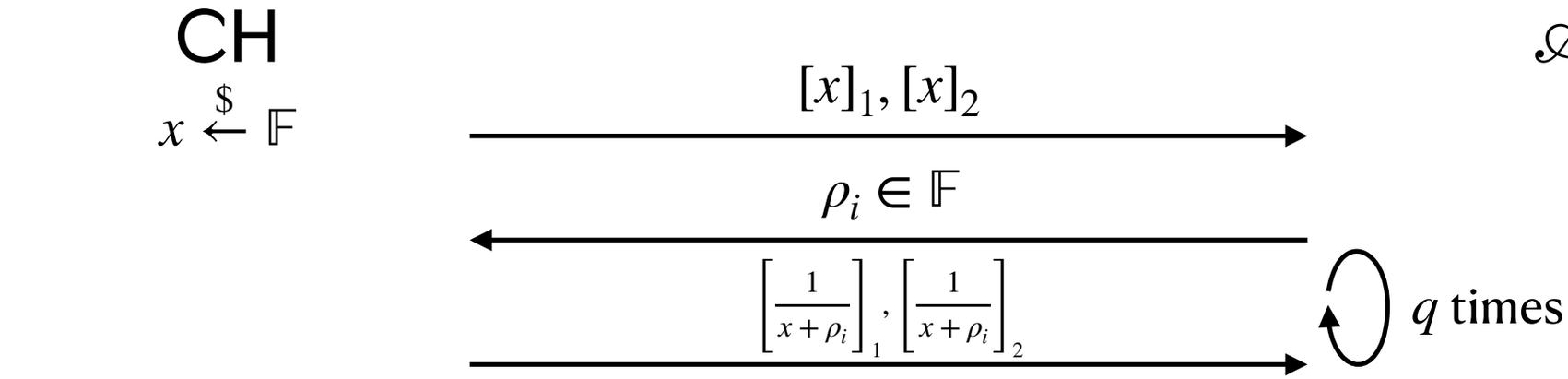
$$\rho \notin \{\rho_i\}_{i\in[q]} \text{ and } u = \left[\frac{1}{x+\rho}\right]_2$$

Credential-based KVCs

Defining and Proving Security

- Usual notion of *key-binding* (slightly modified to account for *unregistered* keys)
- Two new assumptions, (1) q-DistPole: reminiscent of BB signature *unforgeability,* and (2) q-SqPole: cannot compute a quadratic fraction from linear fractions

q-SqPole



$\mathscr{A}$ wins if,

$$u = \left[\frac{1}{(x+\rho_j)^2}\right]_2$$

Credential-based KVCs

Efficiency & Comparison with Prior Work

- In fact, loosely similar to RSA KVaC [AR20] which is instantiated in composite-order groups
  - vs. prime-order groups in our case. Much smaller in practice $\Longrightarrow$ less communication
  - computationally more efficient. No expensive Bezout coefficients calculation
- Cuckoo commitments [FKdP23] constructs KVCs from Vector Commitments generically using cuckoo hashing
  - our construction is *purely* algebraic avoiding non-algebraic operations such as cuckoo hashing
  - we support stateless updates while Cuckoo commitments does not

Second Construction
Dynamic Threshold Encryption

Dynamic Threshold Encryption (DTE) [DP08]

overview

- Public-key encryption with threshold decryption [DF89]
  - Given ciphertext, users can generate *decryption shares,* which can be combined to recover plaintext
- Dynamic selection of threshold and authorized set at encryption time
- In our setting, central authority issues keys using secret (DKG relatively simple and efficient in our case so can be multiple parties in MPC)
- For simplicity, assume users are in $[n]$ but can extend to arbitrary identities using a hash function...
- Define $p_i(X) = \dfrac{1}{X + i}$

$$\mathsf{pk}_i = \left[p_i(x)\right]_1 \in \mathbb{G}_1 \quad \text{and} \quad \mathsf{sk}_i = \left[p_i(x)\right]_2 \in \mathbb{G}_2$$

Dynamic Threshold Encryption (DTE)

preprocessing the encryption key

- Public preprocess encryption key as sum of public keys *not in* the authorized set or in other words, the *unauthorized set*
- For authorized set $S \subseteq [n]$ of size s,

$$\mathsf{ek}_S = \sum_{i \notin S} \mathsf{pk}_i = \left[\sum_{i \notin S} p_i(x) \right]_1$$

- Think of ek_S as a linear relation on $n - s$ *variables*, $p_i(x)$

Dynamic Threshold Encryption (DTE)

Using Linear Independence (2)

$$\mathsf{ek}_S = \sum_{i \notin S} \mathsf{pk}_i = \left[\sum_{i \notin S} p_i(x) \right]_1$$

- Multiply (pairing) with secret key of authorized user $j \in S$,

$$e\left(\mathsf{ek}_S, \mathsf{sk}_j\right) = \left[\sum_{i \notin S} p_i(x) \cdot p_j(x) \right]_T = \left[\sum_{i \notin S} \frac{1}{j-i} \left(p_i(x) - p_j(x) \right) \right]_T$$

$$\text{So, } e\left(\mathsf{ek}_S, \mathsf{sk}_j\right) + \left[\sum_{i \notin S} \frac{1}{j-i} p_j(x) \right]_T = \left[\sum_{i \notin S} \frac{1}{j-i} p_i(x) \right]_T$$

- Which are linearly independent for distinct $j, j' \in S$ by (2)

Dynamic Threshold Encryption (DTE)

Using Linear Independence (2)

$$\mathsf{ek}_S = \sum_{i \notin S} \mathsf{pk}_i = \left[\sum_{i \notin S} p_i(x) \right]_1 \qquad e\left(\mathsf{ek}_S, \mathsf{sk}_j\right) + \left[\sum_{i \notin S} \frac{1}{j-i} p_j(x) \right]_T = \left[\sum_{i \notin S} \frac{1}{j-i} p_i(x) \right]_T$$

- Simplifying,

$$e\left(\mathsf{ek}_S, \mathsf{sk}_j\right) + \sum_{i \notin S} \frac{1}{j-i} \left[p_j(x)\right]_T = \sum_{i \notin S} \frac{1}{j-i} \left[p_i(x)\right]_T$$

$$\implies e\left(\mathsf{ek}_S + \sum_{i \notin S} \frac{1}{j-i} \left[1\right]_1, \mathsf{sk}_j\right) = \sum_{i \notin S} \frac{1}{j-i} \left[p_i(x)\right]_T$$

- So, we obtain linear relations over variables $\left[p_i(x)\right]_T$

- *Intuition*. Every authorized user can contribute one linear relation. Given enough linear relations, we obtain a full-rank system and can solve for some mask that hides the plaintext

Dynamic Threshold Encryption (DTE)

Adding Mask + Generating the Ciphertext

$$\mathsf{ek}_S = \sum_{i \notin S} \mathsf{pk}_i = \left[\sum_{i \notin S} p_i(x) \right]_1 \qquad e\left(\mathsf{ek}_S + \sum_{i \notin S} \frac{1}{j-i} \left[1\right]_1, \mathsf{sk}_j \right) = \sum_{i \notin S} \frac{1}{j-i} \left[p_i(x)\right]_T$$

- Sample a "target" masking key, $\mathsf{tk} = \left[\frac{1}{x-1} = p_{-1}(x) \right]_1$
- We add it to ek_S so the linear relations are also over $\left[p_{-1}(x)\right]_T$
- Randomize the entire linear system so keys are reusable... Sample $r \xleftarrow{\$} \mathbb{F}$
- For message $m \in \mathbb{G}_T$,

$$\mathsf{ct} = \left([r]_1,\ r \cdot (\mathsf{ek}_S + \mathsf{tk}),\ r \cdot \left[p_{-1}(x)\right]_T + m \right)$$

random mask in $\mathbb{G}_1$

to extract linear relation of randomized system

mask recoverable after solving linear system

Dynamic Threshold Encryption (DTE)

Modifying Partial Decryption

$$\mathsf{ek}_S = \sum_{i \notin S} \mathsf{pk}_i = \left[\sum_{i \notin S} p_i(x) \right]_1 \qquad e\left(\mathsf{ek}_S + \sum_{i \notin S} \frac{1}{j-i} \left[1\right]_1, \mathsf{sk}_j \right) = \sum_{i \notin S} \frac{1}{j-i} \left[p_i(x)\right]_T \qquad \mathsf{ct} = \left([r]_1,\ r \cdot (\mathsf{ek}_S + \mathsf{tk}),\ r \cdot \left[p_{-1}(x)\right]_T + m \right)$$

- Use ct_1 and ct_2 instead,

$$e\left(\mathsf{ct}_2 + \left(\sum_{i \notin S} \frac{1}{j-i} + \frac{1}{j+1} \right) \mathsf{ct}_1, \mathsf{sk}_j \right)$$

$$= \left[r \left(\sum_{i \notin S} p_i(x) + p_{-1}(x) + \sum_{i \notin S} \frac{1}{j-i} + \frac{1}{j+1} \right) p_j(x) \right]_T$$

$$= \sum_{i \notin S} \frac{r}{j-i} \left[p_i(x)\right]_T + \frac{r}{j+1} \left[p_{-1}(x)\right]_T$$

Dynamic Threshold Encryption (DTE)

Adding Dummies for Arbitrary Threshold

$$\mathsf{ek}_S = \sum_{i \notin S} \mathsf{pk}_i = \left[\sum_{i \notin S} p_i(x) \right]_1 \qquad e\left(\mathsf{ct}_2 + \left(\sum_{i \notin S} \frac{1}{j-i} + \frac{1}{j+1} \right) \mathsf{ct}_1, \mathsf{sk}_j \right) \qquad \mathsf{ct} = \left([r]_1,\ r \cdot (\mathsf{ek}_S + \mathsf{tk}),\ r \cdot \left[p_{-1}(x) \right]_T + m \right)$$

- Number of variables in system ≡ number of partial decryptions required to unmask message
- How do we go from $n - s + 1$ variables to something that depends on the threshold t?
- We add *dummy* partial fractions to the encryption key
  - Act like *variables* in the linear system
  - Add $s + t$ dummies to get a linear system over $n + t + 1$ variables
- How to decrypt with only t partial decryption shares?
  - Give n secret keys “for free” in the aggregation/combining key ck
- Dummies can be added dynamically at encryption time

Dynamic Threshold Encryption (DTE)

Consolidating Dummies

$$\mathsf{ek}_S = \sum_{i \notin S} \mathsf{pk}_i = \left[\sum_{i \notin S} p_i(x) \right]_1 \qquad e\left(\mathsf{ct}_2 + \left(\sum_{i \notin S} \frac{1}{j-i} + \frac{1}{j+1} \right) \mathsf{ct}_1, \mathsf{sk}_j \right) \qquad \mathsf{ct} = \left([r]_1,\ r \cdot (\mathsf{ek}_S + \mathsf{tk}),\ r \cdot \left[p_{-1}(x) \right]_T + m \right)$$

- As is, need $2n$ dummies to support any S and t
- The identity of the dummies don't matter—we just need an appropriate number of them
  - Borrow trick from recent work by Waters and Wu [WW25] to publish sums of dummies in increasing powers of 2
- Thus, only need $O(\log n)$ public parameters!
- (We ignore dummies henceforth for clarity in this presentation...)

Dynamic Threshold Encryption (DTE)

Share Verification + Reducing Communication

$$\mathsf{ek}_S = \sum_{i \notin S} \mathsf{pk}_i = \left[\sum_{i \notin S} p_i(x) \right]_1 \qquad e\left(\mathsf{ct}_2 + \left(\sum_{i \notin S} \frac{1}{j-i} + \frac{1}{j+1} \right) \mathsf{ct}_1, \mathsf{sk}_j \right) \qquad \mathsf{ct} = \left([r]_1,\ r \cdot (\mathsf{ek}_S + \mathsf{tk}),\ r \cdot \left[p_{-1}(x) \right]_T + m \right)$$

- Verify partial decryption share is valid with respect to user's verification key vk_i (which will end up being equal to pk_i in our case). We modify partial decryption to support this...
- Converts one $\mathbb{G}_T$ element into one $\mathbb{G}_1$ and one $\mathbb{G}_2$ element, reducing communication in practice
- Sample $r_i \xleftarrow{\$} \mathbb{F}$ and output, $\partial_i = \left(\frac{1}{r_i}\mathsf{ct}_1, \frac{1}{r_i}\mathsf{ct}_2, \left[r_i \right]_2, r_i \cdot \mathsf{sk}_i \right)$
- To verify, check ct and randomized sk_i are consistent with randomness r_i and pk_i,

$$e\left(pk_i, \left[r_i \right]_2 \right) \stackrel{?}{=} e\left([1]_1, r_i \cdot \mathsf{sk}_i \right) \textit{ and } e\left(\frac{1}{r_i}\mathsf{ct}_1, \left[r_i \right]_2 \right) \stackrel{?}{=} e\left(\mathsf{ct}_1, [1]_2 \right) \textit{ and } e\left(\frac{1}{r_i}\mathsf{ct}_2, \left[r_i \right]_2 \right) \stackrel{?}{=} e\left(\mathsf{ct}_2, [1]_2 \right)$$

- Can be batched into only 3 pairings total (vs. 6 above)

Dynamic Threshold Encryption (DTE)

Combining/Aggregation

$$\mathsf{ek}_S = \sum_{i \notin S} \mathsf{pk}_i = \left[\sum_{i \notin S} p_i(x) \right]_1 \qquad \partial_i = \left(\frac{1}{r_i}\mathsf{ct}_1, \frac{1}{r_i}\mathsf{ct}_2, \left[r_i\right]_2, r_i \cdot \mathsf{sk}_i \right) \qquad \mathsf{ct} = \left([r]_1,\ r \cdot (\mathsf{ek}_S + \mathsf{tk}),\ r \cdot \left[p_{-1}(x)\right]_T + m \right)$$

- Combiner has ck with *free* secret keys (distinct from real users)
- Derives n more linear relations (it turns out it doesn't need to do n additional pairings for this... just 1!)
- Solves for $r \cdot \left[p_{-1}(x)\right]_T$ using inverse of Cauchy matrix
  - Only need first row of the inverse. Closed-form solution efficiently computable in $\tilde{O}(n)$

Dynamic Threshold Encryption (DTE)

CPA Security with Adaptive Corruptions

- A new assumption *q-Decisional Bilinear Rank-Deficit DH* (*q*-RankDef)
  - Briefly, given $\left[\frac{1}{x+\rho_i}\right]_1$ and *not too many** $\left[\frac{1}{x+\rho_i}\right]_2$ for adversarially chosen ρ_i, and $\left[\frac{1}{x+\rho_t}\right]_1$ for an adversarially chosen *target* ρ_t, $\mathcal{A}$ cannot distinguish the following,

$$\left([r]_1, r \cdot \left[\sum_i \frac{1}{x+\rho_i} + \frac{1}{x+\rho_t}\right]_1, r \cdot \left[\frac{1}{x+\rho_t}\right]_T\right)$$

$$\approx_c$$

$$\left([r]_1, r \cdot \left[\sum_i \frac{1}{x+\rho_i} + \frac{1}{x+\rho_t}\right]_1, \cdot [r']_T\right)$$

(* It has a *deficit amount of fractions in* $\mathbb{G}_2$.)

Dynamic Threshold Encryption (DTE)

CPA Security with Adaptive Corruptions

- $\mathcal{A}$ can pick ρ_i adaptively
- Can prove a strong notion of CPA security wherein $\mathcal{A}$ can corrupt parties adaptively both before and after the encryption key is sampled, and even after the challenge, easily using this assumption
- Indeed, assumption is strong because it is adaptive in nature
- Assumption holds in Bilinear GGM
- Interesting if it's possible to reduce this to a static assumption (like BB signature unforgeability [BB04b])

Dynamic Threshold Encryption (DTE)

CCA-2 Security

- Ciphertext is ElGamal-like
- Can use generic transformations to get CCA-2 security with little overheard
- Ciphertext can also be made *publicly verifiable*—to verify if the ciphertext is *decryptable*
- Two approaches: Groth-Sahai NIZK [GS08] (short CRS + large proof) or QA-NIZK [JR13] (larger CRS + shorter proof)

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Comparison with prior work

	Delerablée and Pointcheval [DP08]	Garg et al. [GKPW24]	Waters and Wu [WW25]	This work
$\|\mathsf{ek}\|$	$O(n)$	$O(1)$	$O(\log n)$	$O(\log s)$
$\|\mathsf{pp}\|$	–	$O(n)$	$O(n \log n)$	$O(\log n)$
$\|\mathsf{pk}\|$	$O(1)$	$O(n)$	$O(n)$	$O(1)$
$\|\mathsf{ct}\|$	$2\|\mathbb{G}_1\| + \|\mathbb{G}_T\|$	$2\|\mathbb{G}_1\| + 7\|\mathbb{G}_2\| + \|G_T\|$	$3\|\mathbb{G}\| + \|\mathbb{G}_T\|$	$2\|\mathbb{G}_1\| + \|\mathbb{G}_T\|$ ($+3\|\mathbb{G}_1\| + \|\mathbb{G}_T\|$ for CCA)
$\|\partial\|$	$3\|\mathbb{G}\| + \|\mathbb{G}_T\|$	$1\|\mathbb{G}_2\|$	$2\|\mathbb{G}\|$	$2(\|G_1\| + \|\mathbb{G}_2\|)$
ciphertext validation?	✓	X	X	✓
$\|\mathsf{ck}\|$	$O(n)$	$O(s)$	$O(s)$	$O(n)$
# of pairings for decryption	$O(t)$	0	3	$O(t)$
# of pairings for verifying share	2	2	2	3
No RO?	X	X	✓	✓
Adaptive?	X	✓	X	✓
Assumption	(static) q-type	generic group	(static) q-type	(adaptive) q-type

Other Constructions

Observing Patterns in Other Primitives

- Hadamard Product Proofs
  - Given commitments to $\vec{u}, \vec{v}, \vec{w} \in \mathbb{F}^n$, we want to prove,

$$\vec{u} \circ \vec{v} = \vec{w}$$

  - Commitment using partial fractions, e.g. $\mathsf{com}_{\vec{u}} = \left[\sum_{i \in [n]} \frac{u_i}{x + i} \right]_1$

    enables proving the relation easily
- Registration-based Encryption
  - algebraically simple construction with concrete savings

ePrint 2025/2081

eprint.iacr.org/2025/2081

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